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1998 J. Phys. A: Math. Gen. 31 10087

(http://iopscience.iop.org/0305-4470/31/50/006)

J. Phys. A: Math. Gen. **31** (1998) 10087–10095. Printed in the UK PII: S0305-4470(98)90624-2

A decomposed and reconstituted peak-time Green function

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Received 7 January 1998

Abstract. A Green function G_n at peak-time t_n decomposes into spherical waves. These are strong in the source vicinity, where they accumulate into a reconstituted G_n that is either relatively weak or comparably strong. Approximations are secured with relative error estimates for dispersive and nondispersive continua. Applications are demonstrated in elasticity, plasma physics, superfluid physics and micropolar elasticity.

1. Introduction

The time-harmonic three-dimensional (3D) Green function [1], which describes a medium's response to a pulsating point source $\delta \boldsymbol{x} e^{-i\omega t}$ takes the waveform $A|\boldsymbol{x}|^{-1} \exp[i(\alpha|\boldsymbol{x}| - \omega t)]$ in classical areas such as acoustics and electrodynamics. The amplitude factor $|\boldsymbol{x}|^{-1}$ indicates an outward spherical attenuation starting from the point source; near this, the wave is relatively strong. Sets of similar waveforms that exhibit the same attenuation factor $|\boldsymbol{x}|^{-1}$ are known to exist in multimode systems such as those found in elasticity [2], plasma physics [3] and, more recently, in microstructured elasticity [4]. When those waves accumulate in the source vicinity wherein they are relatively strong, is the net effect there comparably strong like $|\boldsymbol{x}|^{-1}$ or, perhaps, even stronger? It will be shown within a somewhat general context that, *over a sequence of peak times, the net effect there cannot be stronger* but behaves like $|\boldsymbol{x}|^s$ with $s \in \{-1, 1, 3, 5, \ldots\}$; i.e. *it is either comparably strong or weaker in relation to each individual wave*. The reason for this will become apparent. The order s is determined by neither continuum parameters nor the source frequency ω, but solely by two Laplacian indices of the governing equation. Our analysis will also yield, for dispersive or nondispersive continua, a complete near-source approximation of that net effect with a bound for the relative error. *For nondispersive continua, however, the approximated form is ω-independent and holds not only for small $|\boldsymbol{x}|$* (and any ω) *but also for small $|\omega|$ at any off-source $\boldsymbol{x}$*; moreover, it rectifies another possible misconception conveyed by the individual wave amplitudes (as these can approach infinity once $\omega \to 0$) and, additionally, offers inferences for radiation protection and static-state attainment. The objective, then, is to establish those approximations and to understand how they physically emerge through a reconstitution from waves of a decomposed Green function. This approach is significant because the decomposed version, although correct, becomes deceptive and misleading under the specified circumstances. Applications will be treated.

2. Green function

This investigation concerns a Green function $G(\boldsymbol{x}, t)$ due to the pulsating point source $\delta(\boldsymbol{x}) \cos \omega t \, (\omega \neq 0)$ immersed inside an unbounded 3D continuum that responds in

0305-4470/98/5010087+09\$19.50

accordance with

$$\mathcal{L}(\partial^2/\partial t^2, \nabla^2)G = \mathcal{M}(\partial^2/\partial t^2, \nabla^2)\delta(\boldsymbol{x})\cos\omega t \tag{1}$$

where

$$\mathcal{L}(\partial^2/\partial t^2, \nabla^2) = \sum_{r=0}^{\ell} L_r(\partial^2/\partial t^2)\nabla^{2r} \qquad \ell \geqslant 1, L_\ell \not\equiv 0, L_0 \not\equiv 0 \tag{2}$$

$$\mathcal{M}(\partial^2/\partial t^2, \nabla^2) = \sum_{r=0}^{m} M_r(\partial^2/\partial t^2)\nabla^{2r} \qquad m \geqslant 0, M_m \not\equiv 0 \tag{3}$$

L_r, M_r being polynomials in $\partial^2/\partial t^2$ and possessing uniform coefficients; $m = 0$ corresponds to a Laplacian-free $\mathcal{M} = M_0(\not\equiv 0)$. The situation presented arises, in the most classical sense, in acoustics and electrodynamics with $\mathcal{L} = \partial^2/\partial t^2 - c^2\nabla^2$ (c = wave speed) and $\mathcal{M} = 1$ or $\partial^2/\partial t^2$, depending on the response that G represents. A radiation condition [5–7] is invoked for a time-harmonic state by giving the source an artificial growth $\mathrm{e}^{\epsilon t}(\epsilon > 0)$ which is communicated to the time-harmonic response G, so that this becomes $G^\epsilon = \frac{1}{2}[G_+^\epsilon(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t} + G_-^\epsilon(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega t}]\mathrm{e}^{\epsilon t}$. In terms of a Fourier integral [8, 9] involving even functions

$$\mathcal{M}_\pm^\epsilon(\alpha) = \mathcal{M}(-(\omega \mp \mathrm{i}\epsilon)^2, -\alpha^2) \qquad \mathcal{L}_\pm^\epsilon(\alpha) = \mathcal{L}(-(\omega \mp \mathrm{i}\epsilon)^2, -\alpha^2)$$

transformed from the operators $\mathcal{M}$ and $\mathcal{L}$, the spatial factor [3, 5–7]

$$G_\pm^\epsilon = (2\pi)^{-3}\iiint_{-\infty}^{\infty} \exp(\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x})\frac{\mathcal{M}_\pm^\epsilon(|\boldsymbol{\kappa}|)}{\mathcal{L}_\pm^\epsilon(|\boldsymbol{\kappa}|)}\,\mathrm{d}^3\boldsymbol{\kappa}.$$

To reduce this, we perform a positive rotational transformation on $\boldsymbol{\kappa}$ to get $\boldsymbol{\kappa}\boldsymbol{X} = \boldsymbol{\alpha} = |\boldsymbol{\alpha}|(\sin\Theta\cos\Psi, \sin\Theta\sin\Psi, \cos\Theta)(0 \leqslant \Theta \leqslant \pi, 0 \leqslant \Psi \leqslant 2\pi)$ where $\boldsymbol{X} = (\boldsymbol{x}_1^T, \boldsymbol{x}_2^T, \boldsymbol{x}_3^T)$, a 3×3 rotational matrix whose columns are transposes (signified by T-superscripts) of a right-handed set of three mutually orthogonal unit vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 = \boldsymbol{x}|\boldsymbol{x}|^{-1}$. The Jacobian $\partial\boldsymbol{\alpha}/\partial\boldsymbol{\kappa} = \det\boldsymbol{X} = 1$; also, $|\boldsymbol{\alpha}|^2 = \boldsymbol{\alpha}\boldsymbol{\alpha}^T = \boldsymbol{\kappa}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{\kappa}^T = |\boldsymbol{\kappa}|^2$. Thus

$$\begin{aligned} G_\pm^\epsilon &= (2\pi)^{-3}\int_0^\infty \mathrm{d}|\boldsymbol{\alpha}||\boldsymbol{\alpha}|^2\frac{\mathcal{M}_\pm^\epsilon(|\boldsymbol{\alpha}|)}{\mathcal{L}_\pm^\epsilon(|\boldsymbol{\alpha}|)}\int_0^\pi \mathrm{d}\Theta\sin\Theta\exp(\mathrm{i}|\boldsymbol{\alpha}|x\cos\Theta)\int_0^{2\pi}\mathrm{d}\Psi \\ &= (4\pi^2 x\mathrm{i})^{-1}\int_{-\infty}^{\infty}\alpha\exp(\mathrm{i}\alpha x)\frac{\mathcal{M}_\pm^\epsilon(\alpha)}{\mathcal{L}_\pm^\epsilon(\alpha)}\,\mathrm{d}\alpha \end{aligned} \tag{4}$$

with $x = |\boldsymbol{x}|$. Suppose

$$L_\ell(-\omega^2) \neq 0 \qquad L_0(-\omega^2) \neq 0 \qquad M_m(-\omega^2) \neq 0$$

for our choice of source frequency ω. The dispersion function $\mathring{\mathcal{L}}(\alpha) = \mathcal{L}(-\omega^2, -\alpha^2)$ then has ℓ symmetric pairs of nonvanishing zeros at, say, $\alpha = \alpha_j(\omega), -\alpha_j(\omega)$ $(j = 1, \ldots, \ell)$, which we assume are all real, distinct and nonstationary at the selected frequency ω. These assumptions hold for the classical elasticity, plasma, superfluid and micropolar elasticity applications which will be treated. Obviously, in each jth pair, one zero α_j, say, satisfies $\alpha_j'(\omega) > 0$. It suffices that $\epsilon \gtrsim 0$. Then the integrand in (4), when extended into the complex α-plane, is meromorphic with one set of simple poles at $\alpha \sim \alpha_j(\omega) \mp \mathrm{i}\epsilon\alpha_j'(\omega)$ within the half-space : $\operatorname{Im}\alpha \lessgtr 0$ and another set of simple poles at $\alpha \sim -\alpha_j(\omega) \pm \mathrm{i}\epsilon\alpha_j'(\omega)$ within the conjugate half-space: $\operatorname{Im}\alpha \gtrless 0$. During off-source observations, $x > 0$, corresponding to which, a convergence condition of Jordan's lemma is satisfied within $\operatorname{Im}\alpha > 0$ if

$$\ell \geqslant m + 1$$

which is, henceforth, assumed. Contour integration may then be applied to (4). A radiation conditioned solution is $G(\boldsymbol{x}, t) = \lim_{\epsilon\to 0} G^{\epsilon}$. In this paper we are particularly interested in its value

$$G_n(\boldsymbol{x}) = G(\boldsymbol{x}, t_n) \qquad \text{at any instant } t_n = n\pi|\omega|^{-1} \qquad n = 1, 2, \ldots$$

when the temporal source factor $\cos\omega t$ peaks. Thereupon, residue theory yields

$$G_n = (-1)^n (2\pi x)^{-1} \sum_{j=1}^{\ell} \alpha_j \cos(\alpha_j x)\, \overset{\circ}{\mathcal{M}}(\alpha_j)/\overset{\circ}{\mathcal{L}}{}'(\alpha_j) \tag{5}$$

where $\overset{\circ}{\mathcal{M}}(\alpha) = \mathcal{M}(-\omega^2, -\alpha^2)$. G_n is thus decomposed into ℓ spherical cosine waves that are attenuated like x^{-1}. Conversely, they tend to become singularly strong as $\boldsymbol{x}$ approaches the source. It will, however, transpire that this feature can be deceptive as it need not be imparted to G_n in the source vicinity. That $\alpha_j'(\omega) > 0$ for each wave implies that its group velocity (with which its sustaining energy propagates [5–7]) points radially outwards from the source. Our wave spectrum is multimode if $\ell \geqslant 2$.

Our subsequent interpretation relies on a reconstituted version of G_n, to be derived from (5). First, we apply Taylor's theorem and incorporate (3) to get

$$G_n = (-1)^n (2\pi)^{-1} \sum_{r=0}^{m} (-1)^r M_r(-\omega^2) \Bigg[\sum_{s=0}^{N} (-1)^s x^{2s-1}/(2s)! \sum_{j=1}^{\ell} \alpha_j^{2(r+s)+1}/\overset{\circ}{\mathcal{L}}{}'(\alpha_j)$$
$$+ (-1)^{N+1} x^{2N+1}/(2N+2)! \sum_{j=1}^{\ell} \alpha_j^{2(r+N)+3} (\cos\gamma_j)/\overset{\circ}{\mathcal{L}}{}'(\alpha_j) \Bigg]$$

with $N = \ell - r - 1$ (note that $\ell - 1 \geqslant N \geqslant \ell - m - 1 \geqslant 0$), $0 < |\gamma_j| < |\alpha_j| x$ for each $\alpha_j x$. Now,

$$\sum_{j=1}^{\ell} \alpha_j^{2(r+s)+1}/\overset{\circ}{\mathcal{L}}{}'(\alpha_j) = \tfrac{1}{2} \sum_{j=1}^{\ell} \alpha_j^{2(r+s)+1}/\overset{\circ}{\mathcal{L}}{}'(\alpha_j) + (-\alpha_j)^{2(r+s)+1}/\overset{\circ}{\mathcal{L}}{}'(-\alpha_j)$$
$$= \frac{1}{4\pi \mathrm{i}} \oint_{\Gamma} \frac{\alpha^{2(r+s)+1}}{\overset{\circ}{\mathcal{L}}(\alpha)} \,\mathrm{d}\alpha = \begin{cases} 0 & 0 \leqslant s \leqslant N-1 (\ell \geqslant 2) \\ (-1)^{\ell} (2L_{\ell}(-\omega^2))^{-1} & s = N (\ell \geqslant 1) \end{cases}$$

Γ being any origin-centred, closed circular anticlockwise contour circumscribing all zeros of $\overset{\circ}{\mathcal{L}}(\alpha)$, i.e. all singularities of the meromorphic integrand involved; Γ may therefore be allowed to expand into the infinite domain. Consequently, the reconstituted Green function

$$G_n = \phi_n + \xi_n : \qquad \phi_n = A^{(n)} x^{2(\ell-m)-3} \tag{6}$$

$$A^{(n)} = (-1)^{n+1} M_m(-\omega^2)[4\pi L_{\ell}(-\omega^2)(2(\ell - m - 1))!]^{-1} \tag{7}$$

$$\xi_n = (-1)^n (2\pi)^{-1} \sum_{r=0}^{m} (A_r - B_r) x^{2(\ell-r)-1} \tag{8}$$

$$A_r = (-1)^{\ell} M_r(-\omega^2)/(2(\ell - r))! \sum_{j=1}^{\ell} \alpha_j^{2\ell+1} (\cos\gamma_j)/\overset{\circ}{\mathcal{L}}{}'(\alpha_j) \qquad r = 0, \ldots, m \tag{9}$$

$$B_0 = 0 \qquad B_r = M_{r-1}(-\omega^2)[2L_{\ell}(-\omega^2)(2(\ell - r))!]^{-1} \qquad r = 1, \ldots, m (\geqslant 1). \tag{10}$$

The reconstituted wavefield approximation

$$G_n \sim \phi_n \tag{11}$$

holds, subject to a relative error ξ_n/ϕ_n that can be rendered as small as is tolerable by locating the observation point $\boldsymbol{x}$ sufficiently close to the source, namely, given any positive ν, however small, and any distance $\bar{x}$,

$$\left|\frac{\xi_n}{\phi_n}\right| < \nu \qquad \text{if } 0 < x < \begin{cases} \sqrt{\nu|A^{(n)}|B^{-1}} & m = 0 \\ \min\left(\sqrt{\nu|A^{(n)}|B^{-1}}, \bar{x}\right) & 1 \leqslant m \leqslant \ell - 1 \end{cases} \tag{12}$$

$$B = (2\pi)^{-1}C_0(m=0) \qquad (2\pi)^{-1}\sum_{r=0}^{m}(|B_r| + C_r)\bar{x}^{2(m-r)} \qquad (1 \leqslant m \leqslant \ell - 1) \tag{13}$$

$$C_r = |M_r(-\omega^2)|/(2(\ell - r))!\sum_{j=1}^{\ell}|\alpha_j|^{2\ell+1}/|\overset{\circ}{\mathcal{L}}'(\alpha_j)|. \tag{14}$$

For such a proximity, $G_n = \mathrm{O}(x^{2(\ell-m)-3})$ and, unlike its wave constituents in (5), is therefore *weak* if $\ell \geqslant m+2$ but, like those wave constituents, is *strong* like x^{-1} if $\ell = m+1$. (Note that $2(\ell - m) - 3 \in \{-1, 1, 3, 5, \ldots\}$ since $\ell \geqslant m + 1$, so that G_n cannot be stronger than $\mathrm{O}(x^{-1})$.) This phenomenon exhibited by the net wavefield depends solely on the Laplacian index differential $\ell - m$ between the $\mathcal{L}$ and $\mathcal{M}$ operators of the transmitting continuum. It is independent of all material coefficients as well as the vibration frequency ω. However, the approximated wavefield itself does, according to (6), (7) and (11), depend on those material coefficients found in the M_m, L_ℓ operators and, if these actually involve $\partial^2/\partial t^2$, on ω as well. A necessary but insufficient criterion for the near-field $G_n = \mathrm{O}(x^{2(\ell-m)-3})$ to be weak is that $\ell \geqslant 2$, i.e. it should be multimode. *The near-singular* $\mathrm{O}(x^{-1})$ *effects of its* ℓ *wave constituents are mutually neutralized to some extent through an interaction between their* $\cos(\alpha_j x)$ *sinusoidal factors near the source so as to soften the net near-field when* $\ell \geqslant m + 2$. The associated small relative error is bounded in accordance with (12), regardless of whether ϕ_n is weak or strong near the source.

3. Nondispersive continuum

The approximation secured can also apply beyond the source neighbourhood. To illustrate this, we focus on a *nondispersive continuum* [5]. This is characterized by $\mathcal{L}$ and $\mathcal{M}$ operators that are homogeneous in $\partial^2/\partial t^2$ and ∇^2:

$$\mathcal{L}(\partial^2/\partial t^2, \nabla^2) = \sum_{r=0}^{\ell} L_r^* \nabla^{2r}(\partial/\partial t)^{2(\ell-r)} \qquad L_\ell^* \neq 0 \qquad L_0^* \neq 0 \tag{15}$$

$$\mathcal{M}(\partial^2/\partial t^2, \nabla^2) = \sum_{r=0}^{m} M_r^* \nabla^{2r}(\partial/\partial t)^{2(m-r)} \qquad M_m^* \neq 0 \tag{16}$$

L_r^*, M_r^* being uniform material coefficients. Classical elastic medium and Landau's superfluid, to which our results will be applied in due course, are nondispersive. Classical acoustic and electromagnetic radiation propagate as single-mode wavefields through nondispersive media. Now, the scaled ω-free dispersion function $\overset{*}{\mathcal{L}}(\lambda) = \mathcal{L}(-1, -\lambda^2)$ possesses ℓ symmetric pairs of nonvanishing ω-independent zeros at, say, $\lambda = \lambda_j$, $-\lambda_j (j = 1, \ldots, \ell)$, assumed to be all real and distinct with, say, $\lambda_j > 0$. The radiation conditioned α_j-wavenumbers, each complying with $\alpha_j'(\omega) > 0$ and the assumptions made on it, are then $\alpha_j = \omega\lambda_j (j = 1, \ldots, \ell)$. The group and phase velocities of each radiation conditioned wavefunction are then equal and directed radially outwards. From (5),

$$G_n = (-1)^n \omega^{2(m-\ell+1)} (2\pi x)^{-1} \sum_{j=1}^{\ell} \lambda_j \cos(\omega\lambda_j x) \mathcal{M}(-1, -\lambda_j^2) / \overset{*}{\mathcal{L}}'(\lambda_j) \tag{17}$$

which, since $\ell \geqslant m+1$, apparently suggests that, unless $\ell = m+1$, $|G_n| \to \infty$ as $\omega \to 0$ which, in turn, implies that a stable static state cannot be attained at $\omega = 0$. Such an impression, as we shall soon realize, is false. Thus, while (5) and (17) hold for all $\boldsymbol{x}(\neq \mathbf{0})$ and all $\omega(\neq 0)$, their representations can be misleading for small x and small ω. To interpret correctly, we again employ (6)–(10) and note that $L_r(-\omega^2) = (-1)^{\ell-r}\omega^{2(\ell-r)} L_r^*$, $M_r(-\omega^2) = (-1)^{m-r}\omega^{2(m-r)} M_r^*$. So

$$A^{(n)} = (-1)^{n+1} M_m^* [4\pi L_\ell^* (2(\ell - m - 1))!]^{-1} \tag{18}$$

which is now ω-independent; also, on introducing the parameters

$$B_0^* = 0 \qquad B_r^* = |M_{r-1}^*| [2|L_\ell^*| (2(\ell - r))!]^{-1} \qquad r = 1, \ldots, m \tag{19}$$

$$C_r^* = |M_r^*| / (2(\ell - r))! \sum_{j=1}^{\ell} \lambda_j^{2\ell+1} / |\overset{*}{\mathcal{L}}'(\lambda_j)| \tag{20}$$

$$|\xi_n / \phi_n| \leqslant (2\pi |A^{(n)}|)^{-1} \sum_{r=0}^{m} (B_{m-r}^* + C_{m-r}^*)(|\omega| x)^{2(r+1)} \tag{21}$$

follows and governs the relative error incurred in the ω-independent approximation

$$G_n \sim \phi_n = A^{(n)} x^{2(\ell - m) - 3} \tag{22}$$

for a small $|\omega|x$, e.g. for finite $|\omega|$ and small x (so that G_n is *weak* or *strong* according as $\ell \geqslant m+2$ or $\ell = m+1$ respectively) or for finite x and small $|\omega|$; the same small relative error controlled by (21) applies to both of these cases. We now infer that, *within a nondispersive continuum, (i) a G_n-recorder is almost completely protected from finite-frequency radiation when located sufficiently near the source, (ii)* as $\omega \to 0$ (and, hence, $n\pi = |\omega| t_n \to 0$), $G \to \phi_0$ at every finite $\boldsymbol{x}(\neq \mathbf{0})$ and thus *attains a static vibration-free state*. With reference to (i), at a high ω-frequency, the source peaks rapidly between small time intervals of $\pi|\omega|^{-1}$, and the relative error $|\xi_n/\phi_n|$ is negligible if $0 < x \ll |\omega|^{-1}$, in which event, $\boldsymbol{x}$ receives rapid 'flashes' across an alternating sign sequence of G_n-terms that, otherwise, does not vary with n, is virtually ω-free and virtually the same as the static state G at zero frequency. We shall proceed to consider various applications.

4. Applications

4.1. Elasticity

The displacement $\boldsymbol{u}$ generated by a point-concentrated body force $\boldsymbol{f_0}\delta(\boldsymbol{x})\cos\omega t$ in classical elastic material, characterized by its equivoluminal and dilatational wave speeds a, $c(> a\sqrt{2})$, satisfies [2]

$$\boldsymbol{u_{tt}} = \boldsymbol{f_0}\delta(\boldsymbol{x})\cos\omega t - a^2 \nabla \times (\nabla \times \boldsymbol{u}) + c^2 \nabla\nabla \cdot \boldsymbol{u}.$$

Then

$$\boldsymbol{u} = \boldsymbol{f_0} K_1 + (c^2 - a^2) \nabla\nabla \cdot \boldsymbol{f_0} K_2 \tag{23}$$

where

$$\mathcal{L}_s(\partial^2/\partial t^2, \nabla^2) K_s = \delta(\boldsymbol{x}) \cos\omega t \qquad s = 1, 2$$

an equation of the form (1) with

$$\mathcal{L}_s(\partial^2/\partial t^2, \nabla^2) = \begin{cases} \partial^2/\partial t^2 - a^2\nabla^2 & s=1 \\ (\partial^2/\partial t^2 - a^2\nabla^2)(\partial^2/\partial t^2 - c^2\nabla^2) & s=2 \end{cases}$$

these being homogeneous operators of the type (15) with $\ell = 1(s=1), 2(s=2)$. Classical elastic material is clearly nondispersive. $\overset{*}{\mathcal{L}}_f(\lambda) = \mathcal{L}_s(-1, -\lambda^2)$ has distinct symmetric pairs of real zeros determined by $a^2\lambda^2 = 1(s=1), a^2\lambda^2 = 1 = c^2\lambda^2(s=2)$. It ensues from (6), (18)–(22) that, at each peak instant $t_n = n\pi|\omega|^{-1}$,

$$K_s(\boldsymbol{x}, t_n) = \phi_{s,n} + \xi_{s,n}$$

which, for small $|\omega|x$, is almost ω-independent, being approximately

$$\phi_{s,n} = \begin{cases} (-1)^n(4\pi a^2 x)^{-1} & s=1 \\ (-1)^{n+1}x(8\pi a^2 c^2)^{-1} & s=2 \end{cases}$$

subject to a small $\mathrm{O}((|\omega|x)^2)$ relative error:

$$\left|\frac{\xi_{s,n}}{\phi_{s,n}}\right| \leqslant \begin{cases} \frac{1}{2}(|\omega|x)^2 a^{-2} & s=1 \\ \frac{1}{12}(|\omega|x)^2(a^4+c^4)(ac)^{-2}(c^2-a^2)^{-1} & s=2. \end{cases}$$

$K_1(\boldsymbol{x}, t_n)$ is strong while $K_2(\boldsymbol{x}, t_n)$ is weak near $\boldsymbol{x} = \boldsymbol{0}$; however, both are finite for finite x but small $|\omega|$. Their exact versions are, by (17),

$$K_1(\boldsymbol{x}, t_n) = \frac{(-1)^n \cos X_a}{4\pi x a^2}$$

$$K_2(\boldsymbol{x}, t_n) = (-1)^n \frac{\cos X_a - \cos X_c}{4\pi x \omega^2 (c^2 - a^2)}$$

where $X_a = \omega x a^{-1}$, $X_c = \omega x c^{-1}$. These versions independently confirm that $K_1(\boldsymbol{x}, t_n) = \mathrm{O}(x^{-1})$ while $K_2(\boldsymbol{x}, t_n) = \mathrm{O}(x)$ near the spot where the body force is applied. Each peak-time elastic displacement is then, from (23),

$$\boldsymbol{u}(\boldsymbol{x}, t_n) = \frac{(-1)^n}{4\pi x}\left\{a^{-2}\boldsymbol{x}_3 \times (\boldsymbol{f_0} \times \boldsymbol{x}_3)\cos X_a + c^{-2}\boldsymbol{x}_3(\boldsymbol{f_0}\cdot\boldsymbol{x}_3)\cos X_c \right.$$
$$\left. + \frac{3\boldsymbol{x}_3(\boldsymbol{f_0}\cdot\boldsymbol{x}_3) - \boldsymbol{f_0}}{\omega^2 x^2}(X_a \sin X_a - X_c \sin X_c + \cos X_a - \cos X_c)\right\}$$

($\boldsymbol{x}_3 = \boldsymbol{x}x^{-1}$ being the unit radial vector); in fact, it exhibits that form to which the Stokes elastic displacement [10, 11]

$$(4\pi x)^{-1}\left\{a^{-2}\boldsymbol{x}_3 \times (\boldsymbol{f_0} \times \boldsymbol{x}_3)T(t - xa^{-1}) + c^{-2}\boldsymbol{x}_3(\boldsymbol{f_0}\cdot\boldsymbol{x}_3)T(t - xc^{-1}) \right.$$
$$\left. + x^{-2}[3\boldsymbol{x}_3(\boldsymbol{f_0}\cdot\boldsymbol{x}_3) - \boldsymbol{f_0}]\int_{x/c}^{x/a} \tau T(t-\tau)\,\mathrm{d}\tau\right\}$$

(namely, one caused by an arbitrarily time-dependent concentrated body force $\boldsymbol{f_0}\delta(\boldsymbol{x})T(t)$,) specializes for a sinusoidal temporal factor $T(t) = \cos\omega t$ at each instant when this peaks.

4.2. Plasma physics

An electron particle $q = q_0\delta(\boldsymbol{x}-\boldsymbol{x_q})\cos\omega t$ and a concentrated force $\boldsymbol{f} = \boldsymbol{f_0}\delta(\boldsymbol{x}-\boldsymbol{x_f})\cos\omega t$ in a warm collisionless plasma with thermal speed c_1, light speed $c_2(> c_1)$, plasma frequency $\omega_p(\neq 0)$, unit electronic mass and electronic charge -1 jointly induce a deviation d in the electronic number density, an excited electron velocity $\boldsymbol{v}$, excited electric and magnetic fields $\boldsymbol{e}, \boldsymbol{b}$ that satisfy [3]

$$\omega_p^2\boldsymbol{e} + c_1^2\nabla d + \omega_p^2\boldsymbol{v_t} = -\boldsymbol{f} \qquad \boldsymbol{b_t} = -c_2\nabla\times\boldsymbol{e}$$
$$d_t + \omega_p^2\nabla\cdot\boldsymbol{v} = q_t \qquad \boldsymbol{e_t} = c_2\nabla\times\boldsymbol{b} + \omega_p^2\boldsymbol{v}.$$

Whereupon,

$$\boldsymbol{v_t} = \omega_p^{-2}[-\boldsymbol{f} + c_1^2\omega^2 q_0\nabla K_1(\boldsymbol{x}-\boldsymbol{x_q}, t) - c_1^2\nabla\nabla\cdot\boldsymbol{f_0}K_1(\boldsymbol{x}-\boldsymbol{x_f}, t)]$$
$$+\boldsymbol{f_0}K_2(\boldsymbol{x}-\boldsymbol{x_f}, t) + (c_1^2 - c_2^2)\nabla\nabla\cdot\boldsymbol{f_0}K_3(\boldsymbol{x}-\boldsymbol{x_f}, t)$$

for which $\mathcal{L}_s(\partial^2/\partial t^2, \nabla^2)K_s(\boldsymbol{x}, t) = \delta(\boldsymbol{x})\cos\omega t\,(s = 1, 2, 3)$, another equation covered by (1) with

$$\mathcal{L}_s = \partial^2/\partial t^2 - c_s^2\nabla^2 + \omega_p^2 \qquad s = 1, 2, \mathcal{L}_3 = \mathcal{L}_1\mathcal{L}_2.$$

Assuming $|\omega| > |\omega_p|$, $\mathcal{L}_s(-\omega^2, -\alpha^2)$ releases distinct symmetric pairs of real zeros determined by $c_s^2\alpha^2 = \omega^2 - \omega_p^2(s = 1, 2)$, $c_1^2\alpha^2 = \omega^2 - \omega_p^2 = c_2^2\alpha^2(s = 3)$. Whereupon, (6), (7) and (11)–(14) yield $K_s(\boldsymbol{x}, t_n) = \phi_{s,n} + \xi_{s,n}$ which, for small x, can be approximated by

$$\phi_{s,n} = \begin{cases} (-1)^n(4\pi c_s^2 x)^{-1} & s = 1, 2 \\ (-1)^{n+1}x(8\pi c_1^2c_2^2)^{-1} & s = 3 \end{cases}$$

the relative error incurred being, for an arbitrarily small positive ν,

$$\left|\frac{\xi_{s,n}}{\phi_{s,n}}\right| < \nu \qquad \text{if } 0 < x < \begin{cases} \sqrt{2\nu c_s^2}/\sqrt{\omega^2-\omega_p^2} & s = 1, 2 \\ 2c_1c_2\sqrt{3\nu(c_2^2 - c_1^2)}/\sqrt{(\omega^2-\omega_p^2)(c_1^4 + c_2^4)} & s = 3. \end{cases}$$

4.3. Superfluid physics

Consider Landau's superfluid with uniform equilibrium parameters of entropy S, temperature θ, pressure p, superfluid, normal and total densities ρ_1, ρ_2, $\rho = \rho_1 + \rho_2$. Suppose a fluid source $q = q_0\delta(\boldsymbol{x})\cos\omega t$ creates perturbations, e.g. $\boldsymbol{v_1}, \boldsymbol{v_2}$ in superfluid and normal velocities. The governing equations [12] can be manipulated into the matrix form [13]

$$\boldsymbol{\mathcal{D}}(\partial^2/\partial t^2, \nabla^2)\begin{pmatrix}\boldsymbol{v_1}\\ \boldsymbol{v_2}\end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}^{-1}\begin{pmatrix}\nabla q\\ \boldsymbol{0}\end{pmatrix}$$

where

$$\boldsymbol{\mathcal{D}}(\partial^2/\partial t^2, \nabla^2) = \boldsymbol{\mathcal{A}}\nabla^2 - \boldsymbol{I}\partial^2/\partial t^2$$

$\boldsymbol{I}$ being the identity matrix while $\boldsymbol{\mathcal{A}} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{D}$; here,

$$\boldsymbol{A} = \begin{pmatrix}\rho_1 & \rho_2\\ \rho & 0\end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix}1 & 0\\ 1 & -\rho S\end{pmatrix}$$
$$\boldsymbol{C} = \begin{pmatrix}(\partial\rho/\partial p)_\theta & (\partial\rho/\partial\theta)_p\\ (\partial(\rho S)/\partial p)_\theta & (\partial(\rho S)/\partial\theta)_p\end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix}\rho_1 & \rho_2\\ 0 & \rho S\end{pmatrix}.$$

$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$ are nonsingular; so is $\boldsymbol{C}$ since the Jacobian $\partial(\rho, S)/\partial(p, \theta) = \rho^{-1} \det \boldsymbol{C}$ and is implicitly nonvanishing in the mapping $(\rho, S) \Rightarrow (p, \theta)$. Let

$$\det \boldsymbol{\mathcal{D}} = \mathcal{L}(\partial^2/\partial t^2, \nabla^2) \qquad (\text{adj } \boldsymbol{\mathcal{D}})\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}^{-1} = (\mathcal{M}_{rs})_{r,s=1,2}.$$

Then

$$\boldsymbol{v}_r = q_0 \nabla K_r \qquad r = 1, 2$$

where $\mathcal{L}K_r = \mathcal{M}_{r1}\delta(\boldsymbol{x})\cos\omega t$, which also comes under (1) with $\ell = 2$ since $\det \boldsymbol{\mathcal{A}} \neq 0$, while $0 \leqslant m \leqslant 1$. As $\boldsymbol{\mathcal{D}}$ is homogeneous in $\partial^2/\partial t^2$ and ∇^2, so are $\mathcal{L}$ and every $\mathcal{M}_{rs}$. Landau's superfluid is therefore nondispersive. $\boldsymbol{\mathcal{A}}$ has two eigenvalues $c_\pm^2 : 2c_\pm^2 = c_1^2 + c_2^2 \pm \sqrt{(c_1^2 + c_2^2)^2 - 4\gamma^{-1}c_1^2c_2^2} > 0$, where $\gamma = C_p/C_v (> 1)$ with specific heats $C_p = \theta(\partial S/\partial\theta)_p$, $C_v = \theta(\partial S/\partial\theta)_\rho$, while $c_1^2 = (\frac{\partial p}{\partial \rho})_S = \frac{(\partial S/\partial\theta)_p}{\partial(\rho,S)/\partial(p,\theta)}$, $c_2^2 = \frac{\rho_1 S^2\theta}{\rho_2 C_v} = \frac{\rho_1 S^2(\partial\rho/\partial p)_\theta}{\rho_2 \partial(\rho,S)/\partial(p,\theta)}$; c_1 is the ordinary sound speed in a classical Euler fluid. The relevant dispersion function $\overset{*}{\mathcal{L}}(\lambda) = \mathcal{L}(-1, -\lambda^2) = \det(\boldsymbol{I} - \lambda^2\boldsymbol{A})$ has two distinct symmetric pairs of real zeros determined by $c_+^2\lambda^2 = 1 = c_-^2\lambda^2$. Consequently, the nondispersion results (17)–(22) apply together with their implications.

4.4. *Micropolar elasticity*

This is linked to microstructured elasticity [4], which is relevant to the technology of modern materials, e.g. for the construction and aerospace industries. The micropolar elastodynamic formulation [14] of displacement $\boldsymbol{u}$ and microrotation $\boldsymbol{\psi}$ produced by a concentrated body force $\boldsymbol{f_0}\delta(\boldsymbol{x})\cos\omega t$ can be recast as

$$\boldsymbol{u} = \boldsymbol{f_0}K_1 + \nabla\nabla\cdot\boldsymbol{f_0}K_2 \qquad \boldsymbol{\psi} = \omega_0^2\nabla\times\boldsymbol{f_0}K_3$$

where

$$\mathcal{L}_s(\partial^2/\partial t^2, \nabla^2)K_s = \mathcal{M}_s(\partial^2/\partial t^2, \nabla^2)\delta(\boldsymbol{x})\cos\omega t \qquad s = 1, 2, 3$$
$$\mathcal{M}_1 = \partial^2/\partial t^2 - c_4^2\nabla^2 + 2\omega_0^2 \qquad \mathcal{M}_2 = (c_1^2 - c_2^2)\mathcal{M}_1 + c_3^2\omega_0^2 \qquad \mathcal{M}_3 = 1$$
$$\mathcal{L}_1 = \mathcal{L}_3 = \mathcal{M}_1[\partial^2/\partial t^2 - (c_2^2 + c_3^2)\nabla^2] + c_3^2\omega_0^2\nabla^2 \qquad \mathcal{L}_2 = \mathcal{L}_1[\partial^2/\partial t^2 - (c_1^2 + c_3^2)\nabla^2]$$

which are covered by (1)–(3) with $\ell = 2(s = 1, 3)$, $3(s = 2)$, and $m = 1(s = 1)$, 1 or 0 ($s = 2, c_1 \neq c_2$ or $c_1 = c_2$), $0(s = 3)$; $c_1, \ldots, c_4$ are micropolar elastic wave speeds while $\omega_0^2 = c_3^2\zeta^{-1}$, ζ denoting micro-inertia. Assuming $|\omega| > |\omega_0|\sqrt{2}$, $\mathcal{L}_s(-\omega^2, -\alpha^2)$ has, for $s = 1, 3$, two distinct symmetric pairs of real zeros determined by $\alpha^2 = \alpha_\pm^2 = \frac{1}{2}a^{-1}(b \pm \sqrt{\Delta})(> 0)$ with $a = c_4^2(c_2^2 + c_3^2)$, $b = \omega^2(c_2^2 + c_3^2 + c_4^2) - \omega_0^2(2c_2^2 + c_3^2)$, $\Delta = [\omega^2(c_4^2 - c_2^2 - c_3^2) + \omega_0^2(2c_2^2 + c_3^2)]^2 + 4\omega^2\omega_0^2c_3^2c_4^2$ and, for $s = 2$, three distinct symmetric pairs of real zeros determined by $\alpha^2 = \alpha_\pm^2$, $\omega^2(c_1^2 + c_3^2)^{-1}$. Whereupon, near the point of force application,

$$K_s(\boldsymbol{x}, t_n) = \mathrm{O}(x^{-1})(s = 1) \qquad \mathrm{O}(x) \text{ or } \mathrm{O}(x^3)(s = 2, c_1 \neq c_2 \text{ or } c_1 = c_2)$$
$$\mathrm{O}(x)(s = 3).$$

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